UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
Computer Engineering

ECE 204 *Numerical methods*

Approximating the solution to a quadratic equation

Douglas Wilhelm Harder, LEL, M.Math.
dwharder@uwaterloo.ca
dwharder@gmail.com

Introduction

- In this topic, we will
  - Consider the solutions to the quadratic equation
    - The quadratic formula
  - See how it deals with extreme cases
    - Unfortunately, these tend to be quite common
  - Consider an alternate formula for calculating the roots

Quadratic formula

- This should be easy

$$ax^2 + bx^2 + c = 0$$

  - Of course, the solution is:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic formula

- Let's try this in MATLAB:

```
>> a = 0.0002; b = 57023.0; c = 0.000001;
```

- The actual solutions are:

–0.0000000000175367834031881 87

–285114999.999999999982

```
>> (-b - sqrt( b^2 - 4*a*c ))/(2*a)
    ans =
        -285115000
>> (-b + sqrt( b^2 - 4*a*c ))/(2*a)
    ans =
        0
```

100% relative error

Quadratic formula

- What happened?

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \approx \frac{-b+\sqrt{b^2}}{2a}$$

$$= \frac{-b+|b|}{2a} = 0$$

  - Subtractive cancellation as the formula relies on taking the difference of two near-identical values

Alternative formula

- Let's rationalize the numerator:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \frac{-b \mp \sqrt{b^2 - 4ac}}{-b \mp \sqrt{b^2 - 4ac}} = \frac{b^2 - \left(b^2 - 4ac\right)}{2a\left(-b \mp \sqrt{b^2 - 4ac}\right)}$$

$$= \frac{4ac}{2a\left(-b \mp \sqrt{b^2 - 4ac}\right)}$$

$$= \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$$

$$= \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

Alternative formula

- Let's try this in MATLAB:

```
>> a = 0.0002; b = 57023.0; c = 0.000001;
```

- The actual solutions are:

–0.0000000000175367834031881887

–285114999.999999999982

```
>> -2*c/(b + sqrt( b^2 - 4*a*c ))
   ans =
      -1.753678340318819e-11
>> -2*c/(b - sqrt( b^2 - 4*a*c ))
   ans =
      -Inf
```

∞ relative error

Choosing the appropriate formula

- The formula you learned in secondary school is best for finding the larger root in absolute value
- Our alternative formula is best for finding the smaller root in absolute value
  - If $b > 0$, $b + |b| = 2b$ $\quad \frac{-b-\sqrt{b^2-4ac}}{2a} \quad \frac{-2c}{b+\sqrt{b^2-4ac}}$
  - If $b < 0$, $b - |b| = 2b$ $\quad \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \frac{-2c}{b-\sqrt{b^2-4ac}}$
  - In both cases, we avoid subtractive cancelation by choosing the sign of $\sqrt{b^2-4ac}$ to match the sign of b

Implementation

```
std::pair<double, double> quadratic( double a, double b, double c ) {
  assert( a != 0.0 );

  if ( b == 0.0 ) {
    assert( -a*c >= 0.0 );
    return std::make_pair( std::sqrt( -c/a ), -std::sqrt( -c/a ) );
  } else {
    assert( b*b >= 4.0*a*c );
    double disc{ std::sqrt( b*b - 4.0*a*c ) };

    if ( b > 0 ) {
      return std::make_pair( (-b - disc)/(2.0*a), (-2.0*c)/(b + disc) );
    } else {
      return std::make_pair( (-b + disc)/(2.0*a), (-2.0*c)/(b - disc) );
    }
  }
}
```

C++ Code is provided to demonstrate the straight-forward nature of these algorithms and not required for the examination

Summary

- Following this topic, you now
  – Have reviewed the quadratic formula
  – Understand it is not always ideal numerically
    - The numerator is subject to subtractive cancellation
  – Are aware that an alterative formula works when the standard formula does not
  – Have seen an implementation

References

[1] https://en.wikipedia.org/wiki/Quadratic_equation

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Tazik Shahjahan for pointing out typos.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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